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1999 J. Phys. A: Math. Gen. 32 1675

(http://iopscience.iop.org/0305-4470/32/9/014)

J. Phys. A: Math. Gen. **32** (1999) 1675–1690. Printed in the UK PII: S0305-4470(99)97453-X

Boundary conditions in path integrals from point interactions for the path integral of the one-dimensional Dirac particle

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Received 10 September 1998, in final form 30 November 1998

Abstract. A proposal is given, of how to implement point interactions and boundary conditions in the path integral. The starting point is a path-integral formulation of a Dirac particle in one dimension. The implementation of a point interaction yields, by means of a perturbation expansion, the corresponding Green function for a relativistic particle. In the non-relativistic limit several cases can be distinguished depending on which of the four possibilities of the point interaction has been chosen. By a proper combination of the various possibilities of implementing the point interaction, the whole range of the four-parameter family of boundary conditions for point interactions can be exploited, in the relativistic case as well as in the non-relativistic limit. In addition, making the strength of the point interaction infinitely repulsive yields boundary conditions at finite points on the real line. In particular, Dirichlet and Neumann boundary conditions emerge. The method is illustrated with some examples.

1. Introduction

Boundary conditions are of inherent importance in the set-up of a physical problem. Solving the Schrödinger or the Dirac equation without imposing a proper boundary condition makes no sense and does not even define the relevant Hilbert space. For a well-defined Hilbert space boundary conditions at infinity, i.e. the vanishing of the wavefunctions at infinity, are usually sufficient. This kind of boundary condition is contained in the usual path-integral formulation [1] in a homogeneous space for a regular potential in a natural way. Boundary conditions connected with singular potentials require something different and can be taken into account by using a functional weight formulation [2]. The single-valuedness conditions for the quantum motion on spheres is taken into account by periodic boundary conditions [2]. Boundary conditions at infinity, however, are a very specific idealization and for many physical situations are not appropriate. Typical experimental situations require boundary conditions at a finite distance from the origin, for instance, motion in a half-space [2–7], in a box [7–9] or some boundary condition at a singular point [10, 11]. Here Dirichlet and Neumann boundary conditions come into play as particular cases and can be incorporated into the path integral by considering the infinite strength limit of point interactions. As discussed in, for example, [6, 7, 12–18] point interactions, in turn, can be incorporated into the path integral by, for example, a simple δ-function perturbation. The path integral with this perturbation can be evaluated by the summation of a perturbation expansion, giving in the general case the energy-dependent Green function. The corresponding propagator can be obtained only in specific cases, e.g. for free motion subject to a point interaction. The whole procedure can be repeated to incorporate arbitrarily many point interactions. The limit of infinitely repulsive δ-function perturbations gives Dirichlet boundary conditions at the location of the point interaction. Repeating the procedure, one can state the Green function for a particle in a box, where an otherwise arbitrary well behaved potential may be included.

The incorporation of a boundary condition in a quantum mechanical problem which goes beyond the usual boundary conditions at $\pm\infty$ is not trivial. In [11] a comprehensive approach can be found for the particular case of point interactions: these models are explicitly solvable. Point interactions represent some specific kind of boundary condition for the wavefunction. For instance, the wavefunction is no longer continuous and differentiable continuous, but now has a jump in its first derivative. In [11] this was performed for the free motion, including the usual point interaction in terms of a δ-function, and the so-called δ'-function in one dimension, point interactions in two and three dimensions, and point interactions for the one-dimensional Dirac particle. These models in one dimension belong to a four-parameter family of point interactions [4, 8, 11]. A model with the particular emphasis on the hydrogen atom can be found in [19], including many references. The general feature of the solution for the corresponding Green function of a quantum mechanical model with a point interaction is Krein's formula. It consists of a term representing the Green function of the unperturbed model without a point interaction, and a second term which takes into account the boundary conditions, i.e. the point interaction. However, the model of point interactions is appealing, because in the limit of infinite coupling strength specific boundary conditions emerge, i.e. depending on the chosen point interaction, Dirichlet or Neumann boundary conditions emerge. In a successive way then any combination of boundary conditions can be modelled, and all four parameters of the four-parameter family can be taken into account.

In this contribution I would like to demonstrate, on the one hand, how one can incorporate the four-parameter family of the one-dimensional relativistic point interaction into the path integral, and one the other hand, summarize its applications in non-relativistic quantum mechanics. It generalizes previous attempts to incorporate point interactions into the path integral by giving a prescription as to how to incorporate all four parameters of the four-parameter family into the path-integral formalism. Point interactions are often used to simplify more complicated interactions by a simple solvable model, whether they are quark–quark interactions in elementary particle physics [11] (and references therein), or electron–lattice interactions in solid state physics [11, 20, 21]. The non-relativistic limit includes the usual δ-function perturbation [6, 13, 14, 22, 23], the δ'-function perturbation [11, 17] or more generally the four-parameter family of boundary conditions on the real line [10, 24]. Dirichlet and Neumann boundary conditions follow from limiting cases, where the strength of the point interaction becomes infinitely repulsive.

The relevant one-dimensional Dirac operator has the following form ($\boldsymbol{\sigma}_{x,y,z}$ are the Pauli matrices):

$$D = c\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} \otimes \boldsymbol{\sigma}_x + mc^2 \otimes \boldsymbol{\sigma}_z = \begin{pmatrix} mc^2 & c\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} \\ c\frac{\hbar}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x} & -mc^2 \end{pmatrix} \tag{1}$$

which acts on two-component wavefunctions. Incorporating a potential $\boldsymbol{U}(x)$ we obtain in the usual way for the Hamiltonian in terms of the position and momentum operators

$$H = cp_x \otimes \boldsymbol{\sigma}_x + mc^2 \otimes \boldsymbol{\sigma}_z + \boldsymbol{U}(x). \tag{2}$$

$\boldsymbol{U}(x)$ is in general a 2×2 matrix, of course. Now, we consider a matrix-valued point interaction for the one-dimensional Dirac particle

$$\boldsymbol{U}(x) = (\mathbb{1}g_V + \boldsymbol{\sigma}_z g_S)\,\delta(x-a) \tag{3}$$

where the quantities g_V and g_S correspond to the so-called vector and scalar coupling strength in the weak interaction. We have already written explicitly the δ-potential behaviour of the potential, respectively the point-like behaviour of the potential. However, an arbitrary peak function around $x = a$ with supp $= \{0\}$ also does the job. We must consider the boundary condition for a wavefunction $\psi(x)$ approaching $x = a$ from the left and from the right, respectively. The emerging boundary condition can now be described in the following way, e.g. [25]:

$$\psi(a^+) = P \exp\left(\int_{a^-}^{a^+} K(x)\,\mathrm{d}x\right)\psi(a^-) \tag{4}$$

where $K(x) = (\hbar/\mathrm{i})\boldsymbol{\sigma}_x\big(mc^2 \otimes \boldsymbol{\sigma}_z + U - E\big)$, and where P denotes a Dyson-type ordering operator. This specific form of the potential gives rise to a boundary condition of the spinor wavefunctions at the location $x = a$ of the point interaction according to

$$\psi(a^+) = \exp\left(\frac{\hbar}{\mathrm{i}}\boldsymbol{\sigma}_x(g_V + \boldsymbol{\sigma}_z g_S)\right)\psi(a^-). \tag{5}$$

Evaluating the eigenvalues of the exponentiated term we finally obtain [25],

$$\psi(a^+) = \boldsymbol{\Lambda}\psi(a^-) \qquad \boldsymbol{\Lambda} = \cos\sqrt{g_V^2 - g_S^2}\begin{pmatrix} 1 & -\mathrm{i}\alpha_- \\ -\mathrm{i}\alpha_+ & 1 \end{pmatrix} \tag{6}$$

where one has introduced $\alpha_\pm = (g_V \pm g_S)\tan\sqrt{g_V^2 - g_S^2}\big/\sqrt{g_V^2 - g_S^2}$. From this representation it can be seen that there exists a four-parameter family self-adjoint extension of the corresponding Dirac Hamiltonian with a point interaction. (a) In the case where $g_V = g_S$ we have $\alpha_- = 0$ and $\alpha_+ = 2g_V$. This yields in the non-relativistic limit a δ-function perturbation. (b) In the case where $g_V = -g_S$ we have $\alpha_+ = 0$ and $\alpha_- = 2g_V$. This yields in the non-relativistic limit a δ'-function perturbation. (c) If $|g_V| \neq |g_S|$ and g_V, g_S are real we obtain a matrix according to

$$\Lambda = \begin{pmatrix} \cos\lambda & \mathrm{i}\sin\lambda \\ \mathrm{i}\sin\lambda & \cos\lambda \end{pmatrix}$$

with $\lambda = \sqrt{g_V^2 - g_S^2}$. (d) Finally, we find for $|g_V| \neq |g_S|$ and g_V, g_S imaginary a matrix according to

$$\Lambda = \begin{pmatrix} \cosh\lambda & \sinh\lambda \\ \sinh\lambda & \cosh\lambda \end{pmatrix}$$

with $\lambda = -\mathrm{i}\sqrt{\left|g_V^2 - g_S^2\right|}$ [25–27].

By a proper combination it is therefore possible to cover all the relevant parameters in this family with cases (a) and (b) as the building blocks. The corresponding point perturbations are additive and therefore it is possible to first evaluate a perturbative expansion for one point interaction, and in the second step for the other. Consequently, I can construct by a subsequent consideration of each of the four members of the family the Green function of the entire problem. Taking the non-relativistic limit then gives the corresponding case of the four-parameter family of the boundary conditions at a point on the real line [10, 24]. Some of the results have already been announced in [17].

In the following the technique to achieve this is outlined. I concentrate on the two most important cases of $g_V = g_S$ and $g_V = -g_S$. In the next section I present the path-integral representation for the one-dimensional Dirac particle according to Ichinose and Tamura [28] which is based on an idea of Feynman and Hibbs [1]. I incorporate point interactions and derive the corresponding Green functions, in particular for the particle (or electron) and anti-particle (or positron) component. They turn out to correspond to a δ- and δ'-function perturbation in the non-relativistic limit. In a second step, I calculate combinations of these point interactions, i.e. two point interactions in the particle component, two point interactions in the anti-particle component, and a combination of a point interactions in the particle component and in the anti-particle component. Including off-diagonal point interactions, these are the building blocks for further evaluation in the non-relativistic limit; and, in addition, in the limit for making the strength of the point interactions infinitely repulsive, in order to obtain boundary conditions in half spaces.

In section 3 I give some explicit formulae for the non-relativistic limit, i.e. I consider $c \to \infty$. It turns out that in this limit the point interaction in the particle component yields a one-dimensional δ-potential, and the point interaction in the anti-particle yields a δ'-potential. In particular, for the δ'-potential we obtain automatically in the limiting procedure a regularization prescription. This regularization is necessary due to the singularity of the second derivative of the usual one-dimensional particle Green function G taken at equal arguments. In the following, we obtain by taking the limit of taking the coupling of the point interaction infinitely repulsive Dirichlet and Neumann boundary conditions, respectively. I present some relevant formulae for the corresponding boundary conditions on the real line, i.e. non-relativistic particles moving in half-spaces, respectively for non-relativistic particles moving in boxes. However, not every particular case is considered.

In section 4 I present some of the corresponding propagators explicitly including the case of motion in a box with various boundary conditions. Whereas all the formulae of the preceding sections allow the incorporation of an arbitrary smooth potential and the calculation of the corresponding Green function, only the free-particle case gives the propagator, or Feynman kernel, explicitly. Section 5 contains a summary.

2. Perturbation expansions for a Dirac particle

The general method for the time-ordered perturbation expansion is quite simple. Let us assume that we are given a potential $W(x) = V(x) + \tilde{V}(x)$ in the path integral and suppose that W is so complicated that a direct path integration is not possible. However, the path integral $K^{(V)}$ corresponding to $V(x)$ is assumed to be known. We expand the integrand of the path integral containing $\tilde{V}(x)$ in a perturbation expansion about $V(x)$. The result has a simple interpretation on the lattice: the initial kernel corresponding to V propagates during the short-time interval ϵ unperturbed, then it interacts with $\tilde{V}$ in order to propagate again in another short-time interval ϵ unperturbed, and so on, up to the final state. One then obtains the following series expansion (Feynman and Hibbs [1], Devreese *et al* [13, 29, 30] ($\boldsymbol{x} \in \mathbb{R}^D$):

$$\begin{aligned} K(\boldsymbol{x}'', \boldsymbol{x}'; T) = K^{(V)}(\boldsymbol{x}'', \boldsymbol{x}'; T) + \sum_{n=1}^{\infty} \left(-\frac{\mathrm{i}}{\hbar}\right)^n \left(\prod_{j=1}^{n} \int_{t'}^{t_{j+1}} \mathrm{d}t_j \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{x}_j\right) \\ \times K^{(V)}(\boldsymbol{x}_1, \boldsymbol{x}'; t_1 - t')\, \tilde{V}(\boldsymbol{x}_1)\, K^{(V)}(\boldsymbol{x}_2, \boldsymbol{x}_1; t_2 - t_1) \times \cdots \\ \times \tilde{V}(\boldsymbol{x}_{j-1})\, K^{(V)}(\boldsymbol{x}_j, \boldsymbol{x}_{j-1}; t_j - t_{j-1})\, \tilde{V}(\boldsymbol{x}_j)\, K^{(V)}(\boldsymbol{x}'', \boldsymbol{x}_j; t'' - t_j). \end{aligned} \tag{7}$$

Here I have ordered time as $t' = t_0 < t_1 < t_2 < \cdots < t_{n+1} = t''$ and paid attention to the fact that $K(t_j - t_{j-1})$ denotes the retarded propagator and thus is different from zero only if $t_j \geqslant t_{j-1}$. Several problems in path integration which are definitely non-Gaussian, non-Besselian or non-Legendrian can be addressed by a perturbation expansion approach. Let us mention the incorporation of point interactions (Bauch [22], Goovaerts *et al* [13, 29] and [6, 7, 14–17]) and boundary conditions at finite distances [6, 7]. Also $1/r$- [18, 30] and $1/r^2$-potentials [18] can be treated by means of an exact summation of a perturbation expansion. Particularly in the case of the Coulomb potential this perturbation expansion is an expansion in powers of the coupling of the Coulomb interaction strength [30].

We consider the path-integral representation for the matrix-valued kernel $\boldsymbol{K}^{(V)}(T)$ for the one-dimensional Dirac equation [1, 31–33] ($p_x = -\mathrm{i}\hbar\partial_x$)

$$\begin{aligned}\boldsymbol{K}^{(V)}(x'', x'; T) &= \left\langle x'' \left| \exp\left[-\frac{\mathrm{i}}{\hbar}T\left(c\boldsymbol{\sigma}_x p_x + mc^2\boldsymbol{\sigma}_z + \boldsymbol{V}(x)\right)\right]\right| x'\right\rangle \\ &= \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}\boldsymbol{\nu}(t) \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \boldsymbol{V}(x)\,\mathrm{d}t\right). \end{aligned} \tag{8}$$

$\boldsymbol{V}$ may be a matrix-valued potential, e.g. equation (3). The support property of the measure $\mathcal{D}\boldsymbol{\nu}$ [28] is defined in such a way that the motion it is describing selects paths of N steps each of length $c\epsilon$ ($\epsilon = T/N$ in the lattice representation) that start at x' in the direction α, and end at x'' in the direction β, where α and β take the values 'right' and 'left'. The path integration then is a summation over all reversings of directions [1]. In other words, $\mathcal{D}\boldsymbol{\nu}$ may be regarded as a conditional Wiener measure for the one-dimensional Dirac particle [28]. For $\boldsymbol{V} \equiv 0$ the free motion of a Dirac particle emerges. We introduce the Green function $\boldsymbol{G}^{(V)}(E)$ with its matrix representation

$$\boldsymbol{G}^{(V)}(x'', x'; E) = \begin{pmatrix} G_{11}^{(V)}(x'', x'; E) & G_{12}^{(V)}(x'', x'; E) \\ G_{21}^{(V)}(x'', x'; E) & G_{22}^{(V)}(x'', x'; E) \end{pmatrix}. \tag{9}$$

We first consider a δ-function perturbation in the electron (= '+') component, i.e.

$$\tilde{\boldsymbol{V}} = -\alpha \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta(x-a).$$

We obtain by inserting it into the path integral and summing the perturbation expansion

$$\begin{aligned}\boldsymbol{G}^{(\delta_+)}(x'', x'; E) &= \boldsymbol{G}^{(V)}(x'', x'; E) - \frac{1}{G_{11}^{(V)}(a, a; E) - 1/\alpha} \\ &\quad\times \begin{pmatrix} G_{11}^{(V)}(a, x'; E)\, G_{11}^{(V)}(x'', a; E) & G_{11}^{(V)}(a, x'; E)\, G_{12}^{(V)}(x'', a; E) \\ G_{21}^{(V)}(a, x'; E)\, G_{11}^{(V)}(x'', a; E) & G_{21}^{(V)}(a, x'; E)\, G_{12}^{(V)}(x'', a; E) \end{pmatrix} \end{aligned} \tag{10}$$

$$\begin{aligned} &= \frac{1}{G_{11}^{(V)}(a, a; E) - 1/\alpha} \\ &\times \begin{pmatrix} \begin{vmatrix} G_{11}^{(V)}(x'', x'; E) & G_{11}^{(V)}(a, x'; E) \\ G_{11}^{(V)}(x'', a; E) & G_{11}^{(V)}(a, a; E) - 1/\alpha \end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'', x'; E) & G_{11}^{(V)}(a, x'; E) \\ G_{12}^{(V)}(x'', a; E) & G_{11}^{(V)}(a, a; E) - 1/\alpha \end{vmatrix} \\ \begin{vmatrix} G_{21}^{(V)}(x'', x'; E) & G_{21}^{(V)}(a, x'; E) \\ G_{11}^{(V)}(x'', a; E) & G_{11}^{(V)}(a, a; E) - 1/\alpha \end{vmatrix} & \begin{vmatrix} G_{22}^{(V)}(x'', x'; E) & G_{21}^{(V)}(a, x'; E) \\ G_{12}^{(V)}(x'', a; E) & G_{11}^{(V)}(a, a; E) - 1/\alpha \end{vmatrix} \end{pmatrix}. \end{aligned} \tag{11}$$

Similarly for the positron (= '−') component, i.e.

$$\tilde{\boldsymbol{V}} = \left(4m^2\beta c^2/\hbar^2\right)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\delta(x-b)$$

(the constants have been chosen for convenience, $\tilde{\beta} = 4m^2\beta c^2/\hbar^2$),

$$\begin{aligned}\boldsymbol{G}^{(\delta_-)}(x'',x';E) &= \boldsymbol{G}^{(V)}(x'',x';E) - \frac{1}{\hbar^2/4m^2c^2\beta + G_{22}^{(V)}(b,b;E)} \\ &\quad\times\begin{pmatrix} G_{12}^{(V)}(b,x';E)\,G_{21}^{(V)}(x'',b;E) & G_{12}^{(V)}(b,x';E)\,G_{22}^{(V)}(x'',b;E) \\ G_{22}^{(V)}(b,x';E)\,G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,x';E)\,G_{22}^{(V)}(x'',b;E) \end{pmatrix} \qquad (12)\\ &= \frac{1}{G_{22}^{(V)}(b,b;E)+1/\tilde{\beta}}\end{aligned}$$

$$\times\begin{pmatrix} \begin{vmatrix} G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(b,x';E) \\ G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E)+1/\tilde{\beta} \end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'',x';E) & G_{12}^{(V)}(b,x';E) \\ G_{22}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E)+1/\tilde{\beta} \end{vmatrix} \\ \begin{vmatrix} G_{21}^{(V)}(x'',x';E) & G_{22}^{(V)}(b,x';E) \\ G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E)+1/\tilde{\beta} \end{vmatrix} & \begin{vmatrix} G_{22}^{(V)}(x'',x';E) & G_{22}^{(V)}(b,x';E) \\ G_{22}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E)+1/\tilde{\beta} \end{vmatrix} \end{pmatrix}. \qquad (13)$$

For point interactions in the off-diagonal elements we obtain for

$$\tilde{\boldsymbol{V}} = -(mc\gamma_1/\hbar)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\delta(x-a)$$

(the constants have been chosen for convenience, $\tilde{\gamma}_1 = mc\gamma_1/\hbar$)

$$\begin{aligned}\boldsymbol{G}^{(\gamma_1)}(x'',x';E) &= \boldsymbol{G}^{(V)}(x'',x';E) - G_{21}^{(V)}(a,a;E) - \hbar/mc\gamma_1 \\ &\quad\times\begin{pmatrix} G_{11}^{(V)}(a,x';E)\,G_{21}^{(V)}(x'',a;E) & G_{11}^{(V)}(a,x';E)\,G_{22}^{(V)}(x'',a;E) \\ G_{21}^{(V)}(a,x';E)\,G_{21}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,x';E)\,G_{22}^{(V)}(x'',a;E) \end{pmatrix} \\ &= \frac{1}{G_{21}^{(V)}(a,a;E)-1/\tilde{\gamma}_1}\end{aligned}$$

$$\times\begin{pmatrix} \begin{vmatrix} G_{11}^{(V)}(x'',x';E) & G_{11}^{(V)}(a,x';E) \\ G_{21}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,a;E)-1/\tilde{\gamma}_1 \end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'',x';E) & G_{11}^{(V)}(a,x';E) \\ G_{22}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,a;E)-1/\tilde{\gamma}_1 \end{vmatrix} \\ \begin{vmatrix} G_{21}^{(V)}(x'',x';E) & G_{21}^{(V)}(a,x';E) \\ G_{21}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,a;E)-1/\tilde{\gamma}_1 \end{vmatrix} & \begin{vmatrix} G_{22}^{(V)}(x'',x';E) & G_{21}^{(V)}(a,x';E) \\ G_{22}^{(V)}(x'',a;E) & G_{21}^{(V)}(a,a;E)-1/\tilde{\gamma}_1 \end{vmatrix} \end{pmatrix}. \qquad (14)$$

Similarly for

$$\tilde{\boldsymbol{V}} = -(mc\gamma_2/\hbar)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\delta(x-a)$$

(the constants have been chosen for convenience $\tilde{\gamma}_2 = mc\gamma_2/\hbar$) we obtain

$$\begin{aligned}
\boldsymbol{G}^{(\gamma_1)}(x'', x'; E) &= \boldsymbol{G}^{(V)}(x'', x'; E) - G_{12}^{(V)}(a, a; E) - \hbar/mc\gamma_2 \\
&\quad \times \begin{pmatrix} G_{12}^{(V)}(a, x'; E)\, G_{11}^{(V)}(x'', a; E) & G_{12}^{(V)}(a, x'; E)\, G_{12}^{(V)}(x'', a; E) \\ G_{22}^{(V)}(a, x'; E)\, G_{11}^{(V)}(x'', a; E) & G_{22}^{(V)}(a, x'; E)\, G_{12}^{(V)}(x'', a; E) \end{pmatrix} \\
&= \frac{1}{G_{12}^{(V)}(a, a; E) - 1/\tilde{\gamma}_2} \\
\times & \begin{pmatrix} \begin{vmatrix} G_{11}^{(V)}(x'', x'; E) & G_{12}^{(V)}(a, x'; E) \\ G_{11}^{(V)}(x'', a; E) & G_{12}^{(V)}(a, a; E) - 1/\tilde{\gamma}_2 \end{vmatrix} & \begin{vmatrix} G_{12}^{(V)}(x'', x'; E) & G_{12}^{(V)}(a, x'; E) \\ G_{12}^{(V)}(x'', a; E) & G_{12}^{(V)}(a, a; E) - 1/\tilde{\gamma}_2 \end{vmatrix} \\ \begin{vmatrix} G_{21}^{(V)}(x'', x'; E) & G_{12}^{(V)}(a, x'; E) \\ G_{11}^{(V)}(x'', a; E) & G_{12}^{(V)}(a, a; E) - 1/\tilde{\gamma}_2 \end{vmatrix} & \begin{vmatrix} G_{22}^{(V)}(x'', x'; E) & G_{22}^{(V)}(a, x'; E) \\ G_{12}^{(V)}(x'', a; E) & G_{12}^{(V)}(a, a; E) - 1/\tilde{\gamma}_2 \end{vmatrix} \end{pmatrix}.
\end{aligned} \tag{15}$$

Let us assume for simplicity that the component $G_{11}^{(V)}(E)$ in (9) is known and $\boldsymbol{V}$ is a scalar, then I can derive

$$G_{12}^{(V)}(x, y; E) = \frac{c}{mc^2 - V + E}\, p_x G_{11}^{(V)}(x, y; E) \tag{16}$$

$$G_{22}^{(V)}(x, y; E) = \frac{-1}{mc^2 - V + E}\left(\frac{c^2}{mc^2 - V + E}\, p_x p_y\, G_{11}^{(V)}(x, y; E) + \delta(x - y)\right). \tag{17}$$

From these representations it is easily seen that if $G_{11}^{(V)}(E)$ is of O(1) for $c \to \infty$, $G_{12}^{(V)}(E)$ and $G_{22}^{(V)}(E)$ vanish according to $\propto 1/c$ and $\propto 1/c^2$ for $c \to \infty$, respectively.

In the next step we want to incorporate more than just one relativistic point interaction. Let us first study the case where we have two δ-function perturbations in the electron component with strength α_1, α_2 located at $x = a_1, x = a_2$, respectively. We obtain for the (11)-component

$$G_{11}^{(\delta_+,\delta_+)}(x'', x'; E) = \frac{\begin{vmatrix} G_{11}^{(V)}(x'', x'; E) & G_{11}^{(V)}(x'', a_1; E) & G_{11}^{(V)}(x'', a_2; E) \\ G_{11}^{(V)}(a_1, x'; E) & G_{11}^{(V)}(a_1, a_1; E) - 1/\alpha_1 & G_{11}^{(V)}(a_1, a_2; E) \\ G_{11}^{(V)}(a_2, x'; E) & G_{11}^{(V)}(a_2, a_1; E) & G_{11}^{(V)}(a_2, a_2; E) - 1/\alpha_2 \end{vmatrix}}{\begin{vmatrix} G_{11}^{(V)}(a_1, a_1; E) - 1/\alpha_1 & G_{11}^{(V)}(a_1, a_2; E) \\ G_{11}^{(V)}(a_2, a_1; E) & G_{11}^{(V)}(a_2, a_2; E) - 1/\alpha_2 \end{vmatrix}}. \tag{18}$$

Let us abbreviate

$$D_{\alpha_1\alpha_2}(a_1, a_2; E) = \begin{vmatrix} G_{11}^{(V)}(a_1, a_1; E_n) - 1/\alpha_1 & G_{11}^{(V)}(a_1, a_2; E_n) \\ G_{11}^{(V)}(a_2, a_1; E_n) & G_{11}^{(V)}(a_2, a_2; E_n) - 1/\alpha_2 \end{vmatrix}. \tag{19}$$

The energy levels are determined by the poles of the denominator and given implicitly by

$$D_{\alpha_1\alpha_2}(a_1, a_2; E_n) = 0. \tag{20}$$

For the (12)-component I found

$$G_{12}^{(\delta_+,\delta_+)}(x'',x';E)=\frac{1}{\big[G_{11}^{(V)}(a_2,a_2;E)-1/\alpha_2\big]D_{\alpha_1\alpha_2}(a_1,a_2;E)}$$
$$\times\begin{vmatrix}\begin{vmatrix}G_{12}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'';a_2;E)\\ G_{11}^{(V)}(a_2,x'E) & G_{11}^{(V)}(a_2,a_2;E)-1/\alpha_2\end{vmatrix} & \begin{vmatrix}G_{12}^{(V)}(x'',a_1;E) & G_{12}^{(V)}(a_2,a_1;E)\\ G_{11}^{(V)}(x'',a_2;E) & G_{11}^{(V)}(a_2,a_2;E)-1/\alpha_2\end{vmatrix}\\ \begin{vmatrix}G_{11}^{(V)}(a_1,x';E) & G_{11}^{(V)}(a_2,x';E)\\ G_{11}^{(V)}(a_1,a_2;E) & G_{11}^{(V)}(a_2,a_2;E)-1/\alpha_2\end{vmatrix} & D_{\alpha_1\alpha_2}(a_1,a_2;E)\end{vmatrix}. \tag{21}$$

It is not possible to rewrite the determinant in the determinant into just one 3×3-determinant as for the (11)-component because the number of relevant entries is too large. The other components are similar.

Let us finally combine a point interaction in the electron component with a point interaction in the positron component. The (11)-component has the form

$$G_{11}^{(\delta_+,\delta_-)}(x'',x';E)=\frac{1}{\big[G_{22}^{(V)}(b,b;E)+1/\tilde\beta\big]D_{\alpha\tilde\beta}(a,b;E)}$$
$$\times\begin{vmatrix}\begin{vmatrix}G_{11}^{(V)}(x'',x';E) & G_{12}^{(V)}(x'';b;E)\\ G_{21}^{(V)}(b,x'E) & G_{22}^{(V)}(b,b;E)+1/\tilde\beta\end{vmatrix} & \begin{vmatrix}G_{11}^{(V)}(x'',a;E) & G_{12}^{(V)}(b,a;E)\\ G_{21}^{(V)}(x'',b;E) & G_{22}^{(V)}(b,b;E)+1/\tilde\beta\end{vmatrix}\\ \begin{vmatrix}G_{11}^{(V)}(a,x';E) & G_{12}^{(V)}(b,x';E)\\ G_{21}^{(V)}(a,b;E) & G_{22}^{(V)}(b,b;E)+1/\tilde\beta\end{vmatrix} & D_{\alpha\tilde\beta}(a,b;E)\end{vmatrix}. \tag{22}$$

Here I have abbreviated

$$D_{\alpha\tilde\beta}(a,b;E)=\begin{vmatrix}G_{11}^{(V)}(a,a;E_n)-1/\alpha & G_{11}^{(V)}(a,b;E_n)\\ G_{11}^{(V)}(b,a;E_n) & G_{11}^{(V)}(b,b;E_n)+1/\tilde\beta\end{vmatrix}. \tag{23}$$

The energy levels are determined by the poles of the denominator and given implicitly by $D_{\alpha\tilde\beta}(a,b;E_n)=0$. The other components are similar. The special case that the point interaction is proportional to $\mathbb{1}$ or $\boldsymbol{\sigma}_z$ has been discussed in [27] and is in accordance with our results. In this case the point perturbations of each component contribute additively to the Green function.

3. The non-relativistic limit

We consider the limit $c\to\infty$ in $\boldsymbol{G}^{(\delta_\pm)}$. From the non-relativistic limit of the path integral [1, 28] we know that we have the following limit, cf (16):

$$\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}\boldsymbol{\nu}(t)\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\boldsymbol{V}(x)\,\mathrm{d}t\right)$$
$$\to\begin{pmatrix}1&0\\0&0\end{pmatrix}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot x^2-V(x)\right]\mathrm{d}t\right\} \tag{24}$$

where $V(x)$ is the non-relativistic limit of $\boldsymbol{V}(x)$. In the language of stochastic processes, the measure $\mathcal{D}\boldsymbol{\nu}$ yields in the limit $c\to\infty$ the measure $\mathcal{D}W[x]$ (W being a Wiener process, taken in real time, respectively, Wick rotated) which is interpreted in the usual way as $\mathcal{D}\exp\left((\mathrm{i}/\hbar)\big|\int_{t'}^{t''}\dot x^2\,\mathrm{d}t\right)$, where $\mathcal{D}$ is interpreted as the usual 'Feynman measure' [3]. In the present case we find [11, 26] $\boldsymbol{V}_+(x) \to V_\alpha = -\alpha\delta(x-a)$, and $\boldsymbol{V}_-(x) \to V_\beta = -\beta\delta'(x-a)$, respectively. We find that only the $(1,1)$ component in the Green functions remains finite, all others vanish. Furthermore, we find $G_{11}^{(\delta_+)}(E) \to G^{(\delta)}(E)$ and $G_{11}^{(\delta_-)}(E) \to G^{(\delta')}(E)$, where $G^{(\delta)}(E)$ is the Green function for a potential V with the usual δ-function perturbation in non-relativistic quantum mechanics, and $G^{(\delta')}(E)$ is the Green function for a potential problem V with a δ'-function perturbation, respectively.

3.1. δ-functions

We consider the incorporation of δ-function perturbations, i.e. a δ-function as an additional potential located at $x=a$ with strength γ. Only a closed formula for the corresponding Green function can be stated; an explicit result for the propagator can only be obtained in the simplest or in some exceptional cases, e.g. for $V \equiv 0$. One obtains [14]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}Et/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x) + \gamma\delta(x-a)\right]\mathrm{d}t\right\}$$
$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)\,G^{(V)}(a,x';E)}{G^{(V)}(a,a;E) - 1/\gamma}. \tag{25}$$

Here $G^{(V)}(E)$ denotes the Green function for the unperturbed problem ($\gamma = 0$). Possible bound states are determined by the poles of $G(E)$, i.e. by the equation $G^{(V)}(a,a,E_n) = 1/\gamma$.

3.2. δ'-functions

The next case incorporates a δ'-function perturbation. Taking the non-relativistic limit of $G_{11}^{(\delta_-)}(E)$ one obtains for a δ'-function perturbation in the path integral the representation

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}Et/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x) + \beta\delta'(x-a)\right]\mathrm{d}t\right\}$$
$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}_{,x'}(x'',a;E)\,G^{(V)}_{,x''}(a,x';E)}{\widehat{G}^{(V)}_{,x'x''}(a,a;E) + 1/\beta} \tag{26}$$

$$\widehat{G}^{(V)}_{,xy}(a,a;E) = \left(\frac{\partial^2}{\partial x\partial y}G^{(V)}(x,y;E) - \frac{2m}{\hbar^2}\delta(x-y)\right)\bigg|_{x=y=a}. \tag{27}$$

Note that in the path integral (26) the formal expression '$G_{,xy}(a,a;E)$' is automatically regularized by the removal of an ultraviolet divergence. The divergence in the expression $\partial_x\partial_y G^{(V)}$ for $x-y=a$ is cancelled by the subtraction of the additional δ-function. This regularization prescription is not put in 'by hand' but is *a result*.

3.3. Combination of δ and δ'-functions

From the above considerations it is obvious how to obtain the Green function representation of a combined δ- and δ'-function perturbation. We find

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x) + \alpha\delta(x-a) + \beta\delta'(x-b)\right]\mathrm{d}t\right\}$$
$$= \frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}_{,x'}(x'',b;E) & G^{(V)}(x'',a;E) \\ G^{(V)}_{,x''}(b,x';E) & \widehat{G}^{(V)}_{,x'x''}(b,b;E)+1/\beta & G^{(V)}_{,x''}(b,a;E) \\ G^{(V)}(a,x';E) & G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E)-1/\alpha \end{vmatrix}}{\begin{vmatrix} \widehat{G}^{(V)}_{,x'x''}(b,b;E)+1/\beta & G^{(V)}_{,x''}(b,a;E) \\ G^{(V)}_{,x'}(a,b;E) & G^{(V)}(a,a;E)-1/\alpha \end{vmatrix}}. \tag{28}$$

Setting $a = b$ yields a special case (of boundary conditions).

3.4. Dirichlet boundary conditions

The case of (Dirichlet) boundary conditions, respectively the motion in a half-space, have been addressed by several authors in order to develop a method to incorporate them into the path integral, e.g. Barut and Duru [34], Clark *et al* [5], Carreau [4], Grosche [6, 7], and Janke and Kleinert [9]. In our formalism Dirichlet boundary conditions are obtained when we consider in (25) the limit $\gamma \to -\infty$. This has the consequence that an impenetrable wall appears at $x = a$. The result then is for the motion in the half-space $x > a$, say, [6, 7]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(D)}_{(x>a)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$
$$= G^{(V)}(x'',x';E) - \frac{G^{(V)}(x'',a;E)\,G^{(V)}(a,x';E)}{G^{(V)}(a,a;E)}. \tag{29}$$

Possible bound states are determined by the poles of $G(E)$, i.e. by the equation $G^{(V)}(a,a,E_n) = 0$. Furthermore, for the motion inside a box with boundaries at $x = a$ and $x = b$ and Dirichlet boundary conditions on both sides one obtains $(a < x < b)$ [6–8]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}^{(DD)}_{(a<x<b)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$
$$= \frac{\begin{vmatrix} G^{(V)}(x'',x';E) & G^{(V)}(x'',b;E) & G^{(V)}(x'',a;E) \\ G^{(V)}(b,x';E) & G^{(V)}(b,b;E) & G^{(V)}(b,a;E) \\ G^{(V)}(a,x';E) & G^{(V)}(a,b;E) & G^{(V)}(a,a;E) \end{vmatrix}}{\begin{vmatrix} G^{(V)}(b,b;E) & G^{(V)}(b,a;E) \\ G^{(V)}(a,b;E) & G^{(V)}(a,a;E) \end{vmatrix}}. \tag{30}$$

3.5. Neumann boundary conditions

In an obvious way we can also obtain a path-integral representation in the half-space $x > a$, say, with Neumann boundary conditions at $x = a$ by letting $\beta \to -\infty$ in (26) [16, 17]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}t\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(N)}_{(x>a)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$
$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}_{,x'}(x'', a; E)\, G^{(V)}_{,x''}(a, x'; E)}{\widehat{G}^{(V)}_{,x'x''}(a, a; E)}. \tag{31}$$

The same procedure as for the motion in a box $a < x < b$ with Dirichlet boundary conditions at both boundaries, can be applied for Neumann boundary conditions at both boundaries

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(NN)}_{(a<x<b)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$
$$= \frac{\begin{vmatrix} G^{(V)}(x'', x'; E) & G^{(V)}_{,x'}(x'', b; E) & G^{(V)}_{,x'}(x'', a; E) \\ G^{(V)}_{,x''}(b, x'; E) & \widehat{G}^{(V)}_{,x'x''}(b, b; E) & G^{(V)}_{,x'x''}(b, a; E) \\ G^{(V)}_{,x''}(a, x'; E) & G^{(V)}_{,x'x''}(a, b; E) & \widehat{G}^{(V)}_{,x'x''}(a, a; E) \end{vmatrix}}{\begin{vmatrix} \widehat{G}^{(V)}_{,x'x''}(b, b; E) & G^{(V)}_{,x'x''}(b, a; E) \\ G^{(V)}_{,x'x''}(a, b; E) & \widehat{G}^{(V)}_{,x'x''}(a, a; E) \end{vmatrix}}. \tag{32}$$

Similarly, we obtain for Dirichlet boundary conditions at $x = a$, and Neumann boundary conditions for $x = b$ in the box $a < x < b$

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}^{(DN)}_{(a<x<b)}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(x)\right]\mathrm{d}t\right\}$$
$$= \frac{\begin{vmatrix} G^{(V)}(x'', x'; E) & G^{(V)}_{,x'}(x'', b; E) & G^{(V)}(x'', a; E) \\ G^{(V)}_{,x''}(b, x'; E) & \widehat{G}^{(V)}_{,x'x''}(b, b; E) & G^{(V)}_{,x''}(b, a; E) \\ G^{(V)}(a, x'; E) & G^{(V)}_{,x'}(a, b; E) & G^{(V)}(a, a; E) \end{vmatrix}}{\begin{vmatrix} \widehat{G}^{(V)}_{,x'x''}(b, b; E) & G^{(V)}_{,x''}(b, a; E) \\ G^{(V)}_{,x'}(a, b; E) & G^{(V)}(a, a; E) \end{vmatrix}}. \tag{33}$$

Radial boxes and rings can be taken into account as well, and potentials with absolute value dependence by combining the results for Dirichlet and Neumann boundary conditions, i.e. [17]

$$\frac{\mathrm{i}}{\hbar}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{\mathrm{i}ET/\hbar}\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2 - V(|x|)\right]\mathrm{d}t\right\}$$
$$= G^{(V)}(x'', x'; E) - \frac{G^{(V)}(x'', 0; E)\, G^{(V)}(0, x'; E)}{2G^{(V)}(0, 0; E)}$$
$$- \frac{G^{(V)}_{,x'}(x'', 0; E)\, G^{(V)}_{,x''}(0, x'; E)}{2\widehat{G}^{(V)}_{,x'x''}(0, 0; E)}. \tag{34}$$

4. Examples

It is not possible, except in the simplest examples, to explicitly state the propagator in closed form. Generally only the free particle case can be treated.

4.1. Relativistic point interaction

We consider the unperturbed free particle; the explicit expression for $\boldsymbol{G}^{(0)}(E)$ has the form [11]

$$\boldsymbol{G}^{(0)}(x'',x';E)=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta & \mathrm{sign}(x''-x')\\ \mathrm{sign}(x''-x') & 1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|} \tag{35}$$

where $\zeta=\big(E+mc^2\big)/ck\hbar$, $ck\hbar=\sqrt{E^2-m^2c^4}$. This yields for a δ-function perturbation in the electron component,

$$\begin{aligned}\boldsymbol{G}^{(\delta_+)}(x'',x';E)&=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta & \mathrm{sign}(x''-x')\\ \mathrm{sign}(x''-x') & 1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}\\ &\quad-\frac{\alpha\mathrm{e}^{\mathrm{i}k(|x''-a|+|a-x'|)}}{4c\hbar(c\hbar-\mathrm{i}\alpha\zeta/2)}\begin{pmatrix}\zeta^2 & \zeta\,\mathrm{sign}(x''-a)\\ \zeta\,\mathrm{sign}(a-x') & \mathrm{sign}(x''-a)\,\mathrm{sign}(a-x')\end{pmatrix}.\end{aligned} \tag{36}$$

For $[\alpha]>0$ there is one bound state with energy $E=mc^2\big(1-\lambda^2\big)\big/\big(1+\lambda^2\big)$ ($\lambda=\alpha/2c\hbar$). Similarly, for a δ-function perturbation in the positron component

$$\begin{aligned}\boldsymbol{G}^{(\delta_-)}(x'',x';E)&=\frac{\mathrm{i}}{2c\hbar}\begin{pmatrix}\zeta & \mathrm{sign}(x''-x')\\ \mathrm{sign}(x''-x') & 1/\zeta\end{pmatrix}\mathrm{e}^{\mathrm{i}k|x''-x'|}\\ &\quad+\frac{2m^2\beta\mathrm{e}^{\mathrm{i}k(|x''-a|+|a-x'|)}}{\hbar(2\hbar^3+4\mathrm{i}m^2c\beta/\zeta)}\begin{pmatrix}\mathrm{sign}(x''-a)\,\mathrm{sign}(a-x') & \mathrm{sign}(a-x')/\zeta\\ \mathrm{sign}(x''-a)/\zeta & 1/\zeta^2\end{pmatrix}.\end{aligned} \tag{37}$$

For $[\beta]>0$ there is one bound state with energy $E=-mc^2\big(1-\lambda^2\big)\big/\big(1+\lambda^2\big)$ ($\lambda=2m^2c\beta/\hbar^3$).

4.2. δ-function

Let us consider a simple δ-function potential in the path integral. We obtain the solution [12–14, 22, 23, 35–37]

$$\begin{aligned}\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2+\gamma\delta(x)\right]\mathrm{d}t\right\}&=\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left[\frac{\mathrm{i}m}{2\hbar T}(x''-x')^2\right]\\ &\quad+\frac{m\gamma}{2\hbar^2}\exp\left(-\frac{m\gamma}{\hbar^2}(|x''-a|+|x'-a|)+\frac{\mathrm{i}}{\hbar}\frac{m\gamma^2}{2\hbar^2}T\right)\\ &\quad\times\mathrm{erfc}\left[\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\left(|x''-a|+|x'-a|-\frac{\mathrm{i}}{\hbar}\gamma T\right)\right].\end{aligned} \tag{38}$$

Some more examples have been investigated in [14], and the case of the harmonic oscillator with a δ-function in [38].

4.3. δ'-function

Let us consider a δ'-function potential in the path integral (and the notion δ'-function should not to be taken too literally [11]. We obtain the solution [10, 17]

$$\begin{aligned}&\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\tfrac{1}{2}m\dot{x}^2+\beta\delta'(x-a)\right]\mathrm{d}t\right\}\\ &\qquad=\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left(\frac{\mathrm{i}m}{2\hbar T}|x''-x'|^2\right)+\mathrm{sign}(x''-a)\,\mathrm{sign}(x'-a)\\ &\qquad\quad\times\left(\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\exp\left[\frac{\mathrm{i}m}{2\hbar T}(|x''-a|+|x'-a|)^2\right]\mathrm{sign}(x''-a)\,\mathrm{sign}(x'-a)\right.\end{aligned}$$

$$+\frac{\hbar^2}{2m\beta}\exp\left[-\frac{\hbar^2}{m\beta}(|x''-a|+|x'-a|)+\frac{\mathrm{i}}{\hbar}\frac{\hbar^6}{2m^3\beta^2}T\right]$$
$$\times\operatorname{erfc}\left\{\sqrt{\frac{m}{2\mathrm{i}\hbar T}}\left[(|x''-a|+|x'-a|)-\frac{\mathrm{i}\hbar^3 T}{m^2\beta}\right]\right\}\Bigg) \tag{39}$$
$$=\frac{\hbar^2}{m\beta}\exp\left[-\frac{\hbar^2}{m\beta}(|x''-a|+|x'-a|)+\frac{\mathrm{i}}{\hbar}\frac{\hbar^6}{2m^3\beta^2}T\right]$$
$$\times\operatorname{sign}(x''-a)\operatorname{sign}(x'-a)$$
$$+\frac{1}{2\pi}\int_{\mathbb{R}}\mathrm{d}p\,\exp\left(-\mathrm{i}\frac{p^2\hbar}{2m}T\right)\Bigg(\sin px'\sin px''+\cos px'\cos px''$$
$$+\frac{\mathrm{i}mp\beta/\hbar^2}{1+\mathrm{i}pm\beta/\hbar^2}\mathrm{e}^{\mathrm{i}p(|x'-a|+|x''-a|)}\operatorname{sign}(x''-a)\operatorname{sign}(x'-a)\Bigg). \tag{40}$$

4.4. Motion in a box: Dirichlet–Dirichlet boundary conditions

Let us consider free motion in a box with Dirichlet boundary conditions at $x=-b$ and $x=b$. The general method gives for the Green function ($\kappa=\sqrt{-2mE}/\hbar$)

$$G^{(DD)}(x'',x';E)=\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\cosh[\kappa(|x''-x'|-2b))]-\cosh[\kappa(x''+x')]}{\sinh(2\kappa b)}. \tag{41}$$

The energy spectrum follows from the poles of the Green function yielding

$$E_n=\frac{\hbar^2}{2m}\frac{\pi^2n^2}{4b^2}\qquad n\in\mathbb{N}. \tag{42}$$

By means of the Laplace transformation pair [39], p 224,

$$\Theta_3\left(\frac{1}{2}+\frac{x}{2l}\middle|\frac{\mathrm{i}\pi\tau}{l^2}\right)\quad\Leftrightarrow\quad\frac{l}{\sqrt{s}}\frac{\cosh\left(z\sqrt{s}\right)}{\sinh\left(l\sqrt{s}\right)}\qquad |x|<l \tag{43}$$

I obtain for the propagator the following representations:

$$K^{(DD)}(x'',x';T)=\sqrt{\frac{m}{2\pi\mathrm{i}\hbar T}}\sum_{n\in\mathbb{Z}}\left\{\exp\left[\frac{\mathrm{i}m}{2\hbar T}(x''-x'+4nb)^2\right]\right.$$
$$\left.-\exp\left[\frac{\mathrm{i}m}{2\hbar T}(x''+x'+2(2n+1)b)^2\right]\right\} \tag{44}$$
$$=\frac{1}{4b}\left[\Theta_3\left(\frac{|x''-x'|}{4b}\middle|-\frac{\pi\hbar T}{8mb^2}\right)-\Theta_3\left(\frac{x''+x'}{4b}+\frac{1}{2}\middle|-\frac{\pi\hbar T}{8mb^2}\right)\right] \tag{45}$$
$$=\frac{1}{b}\sum_{n=1}^{\infty}\exp\left(-\mathrm{i}\hbar T\frac{\pi^2n^2}{8mb^2}\right)\sin\left[\frac{\pi n}{2b}(x'+b)\right]\sin\left[\frac{\pi n}{2b}(x''+b)\right] \tag{46}$$

and I have used some properties of the Jacobi theta function $\Theta_3(z|q)$.

4.5. Motion in a box: Neumann–Neumann boundary conditions

Let us consider as the next example free motion in a box with Neumann boundary conditions at $x=-b$ and $x=b$. The general method gives for the Green function ($\kappa=\sqrt{-2mE}/\hbar$)

$$G^{(NN)}(x'',x';E)=\frac{1}{\hbar}\sqrt{-\frac{m}{2E}}\frac{\cosh[\kappa(|x''-x'|-2b))]+\cosh[\kappa(x''+x')]}{\sinh(2\kappa b)}. \tag{47}$$

The energy spectrum follows from the poles of the Green function, yielding

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{4b^2} \qquad n \in \mathbb{N}_0. \tag{48}$$

By means of the same Laplace transformation pair as before I obtain for the propagator the following representations ($\epsilon_0 = 1$, $\epsilon_n = 1$, $n \in \mathbb{N}_0$):

$$K^{(NN)}(x'', x'; T) = \sqrt{\frac{m}{2\pi \mathrm{i}\hbar T}} \sum_{n\in\mathbb{Z}} \left\{ \exp\left[\frac{\mathrm{i}m}{2\hbar T}(x'' - x' + 4nb)^2\right] + \exp\left[\frac{\mathrm{i}m}{2\hbar T}(x'' + x' + 2(2n+1)b)^2\right]\right\} \tag{49}$$

$$= \frac{1}{4b}\left[\Theta_3\left(\frac{|x''-x'|}{4b}\middle| -\frac{\pi\hbar T}{8mb^2}\right) + \Theta_3\left(\frac{x''+x'}{4b} + \frac{1}{2}\middle| -\frac{\pi\hbar T}{8mb^2}\right)\right] \tag{50}$$

$$= \frac{1}{2b}\sum_{n=0}^{\infty} \epsilon_n \exp\left(-\mathrm{i}\hbar T \frac{\pi^2 n^2}{8mb^2}\right) \cos\left[\frac{\pi n}{2b}(x'+b)\right] \cos\left[\frac{\pi n}{2b}(x''+b)\right]. \tag{51}$$

4.6. *Motion in a box: Dirichlet–Neumann boundary conditions*

Let us finally consider free motion in a box with Dirichlet boundary conditions at $x = -b$ and Neumann boundary conditions at $x = b$. The general method gives for the Green function ($\kappa = \sqrt{-2mE}/\hbar$)

$$G^{(DN)}(x'', x'; E) = -\frac{1}{\hbar}\sqrt{-\frac{m}{2E}} \frac{\sinh[\kappa(|x''-x'| - 2b))] - \sinh[\kappa(x''+x')]}{\cosh(2\kappa b)}. \tag{52}$$

The energy spectrum follows from the poles of the Green function, yielding

$$E_n = \frac{\hbar^2}{2m}\frac{\pi^2 \left(n+\frac{1}{2}\right)^2}{4b^2} \qquad n \in \mathbb{N}_0. \tag{53}$$

By means of the Laplace transformation pair [39], p 224,

$$\Theta_2\left(\frac{1}{2} + \frac{x}{2l}\middle| \frac{\mathrm{i}\pi\tau}{l^2}\right) \quad \Leftrightarrow \quad -\frac{l}{\sqrt{s}}\frac{\sinh\left(z\sqrt{s}\right)}{\cosh\left(l\sqrt{s}\right)} \qquad |x| < l \tag{54}$$

I obtain for the propagator the following representations:

$$K^{(DN)}(x'', x'; T) = \sqrt{\frac{m}{2\pi \mathrm{i}\hbar T}} \sum_{n\in\mathbb{Z}} (-1)^n \left\{ \exp\left[\frac{\mathrm{i}m}{2\hbar T}(x'' - x' + 4nb)^2\right] - \exp\left[\frac{\mathrm{i}m}{2\hbar T}(x'' + x' + 2(2n+1)b)^2\right]\right\} \tag{55}$$

$$= \frac{1}{4b}\left[\Theta_2\left(\frac{|x''-x'|}{4b}\middle| -\frac{\pi\hbar T}{8mb^2}\right) - \Theta_2\left(\frac{x''+x'}{4b} + \frac{1}{2}\middle| -\frac{\pi\hbar T}{8mb^2}\right)\right] \tag{56}$$

$$= \frac{1}{b}\sum_{n=0}^{\infty} \exp\left(-\mathrm{i}\hbar T \frac{\pi^2 \left(n+\frac{1}{2}\right)^2}{8mb^2}\right) \sin\left[\frac{\pi\left(n+\frac{1}{2}\right)}{2b}(x'+b)\right] \times \sin\left[\frac{\pi\left(n+\frac{1}{2}\right)}{2b}(x''+b)\right]. \tag{57}$$

5. Summary

In this contribution I have shown the various features of point interactions in the path integral. I have started from the path-integral representation of the one-dimensional Dirac particle with a point interaction incorporated. By considering two kinds of point interactions I have been able to derive the corresponding Green functions by means of an exact summation of a perturbation expansion which served as the building blocks for further investigation. In the general case of multiple point interactions it seems not to be possible to derive a simple determinant expression as for the non-relativistic case. One obtains a matrix whose entries are determinants within determinants, etc. In the non-relativistic limit they corresponded to a δ- and a δ'-function perturbation, respectively. Of course, all the corresponding Green functions represent Krein's formula. The path-integral approach shows in a nice way that in comparison to the Schrödinger equation approach a properly defined (and regularized, if necessary) path integral provides a global picture of the problem. However, whereas Krein's formula is usually derived by means of functional analytical methods [11], we obtain then by a summation of a perturbation expansion. The necessary ingredients are the path-integral formulation of the one-dimensional Dirac particle, including its non-relativistic limit, and the knowledge of the Green function for the one-dimensional Dirac particle. No further assumptions have been made. The outcome of the regularization scheme, in particular, for the δ'-function perturbation is quite satisfactory, and it shows that the 'sum over paths' in an exact summation of a perturbation expansion offers possibilities for the solution of problems which go beyond the usual 'Gaussian sum over paths'. I could derive the general feature of the Green function for the four-parameter family point interaction for the one-dimensional Dirac particle thus providing a unified approach. Of course, the four formulae (11), (13)–(15) can be combined yielding more complicated point interactions, respectively boundary conditions for the relativistic and the non-relativistic case as well. Considering the non-relativistic limit, the corresponding (parametrized) point interactions for a one-dimensional Schrödinger particle can been derived. The limit of infinitely repulsive point interactions has yielded Dirichlet and Neumann boundary conditions, respectively. I have demonstrated the technique by several examples, and for the cases where the propagator could be stated explicitly. The presented approach generalizes previous attempts by a systematic description of the incorporation of boundary conditions at finite distances form the origin. It can include Dirichlet and Neumann boundary conditions for Cartesian or radial boxes and rings, shell interactions in radial problems, by taking into account all parameters of the four-parameter family in a successive way. The final result in all cases is the Green function of the problem, from which the bound state energy levels and wavefunctions can be determined in a unique way: the bound states are derived form the poles of the Green function, e.g. $D_{\alpha_1\alpha_2}(E_n) = 0$ in (20) and the scattering states by the cut in the Green function. Therefore it is possible to incorporate general boundary conditions in the path integral in an explicit way by means of a singular perturbation.

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